PISTON THEORY APPLIED TO WING-BODY CONFIGURATIONS: A REVIEW OF THE MATHEMATICAL BASIS

Marius-Corne Meijer
University of Pretoria
Roper Street, Pretoria,
South Africa
mariuscmeijer@gmail.com

Abstract. The theoretical basis and key developments of piston theory are reviewed, with a focus on the mathematical development behind deriving piston theory from the Euler equations. Perturbations relative to an established transverse flow are also considered in forming a mathematical basis for local piston theory. Various conditions are considered under which the validity of assumptions in the mathematical development deteriorate, and these are used in discussing the validity of applying piston theory to wing-body configurations. A brief discussion on the extension to viscous flows is also given.

Keywords. Piston Theory, Unsteady Analogy, Interference, Aeroelasticity.

1 Introduction

Piston theory [1] is a widely-used method for predicting the aerodynamic loading on flight-vehicle surfaces at supersonic and hypersonic speeds in aeroelastic analysis. The appeal of piston theory to the aeroelastician is that surface pressures are related by a point-function relationship to the local surface motion and inclination; this renders the method computationally inexpensive. The method is also conceptually simple and intuitive, and as a result, the mathematical basis for piston theory has been largely neglected in Western literature, such as in the original paper by Lighthill [2]. This may be contrasted with a more mathematically rigorous development in Russian literature, as in the case of Il'yushin [3]. The neglect of a rigorous development in Western literature has led to inconsistent application of piston theory, with various formulations being used under the common label of "piston theory", as reviewed in [4].

Several developments and extensions to piston theory and its application may be noted, with the method continuing to receive attention [5] due to its low computational cost. A significant contribution to the continued interest in and application of piston theory is its use to model perturbations about an existing mean steady flow, which has received the label of "local piston theory" [6]. Recent literature shows the increased application of local piston theory to steady Euler solutions in hypersonic aeroelastic analysis, with excellent agreement typically being obtained [7] with full unsteady Euler computations at a fraction of the computational cost.

Much like in the case of classical piston theory, the mathematical basis of local piston theory has not received attention in literature. The typically good prediction obtained by local piston theory has led to its broader application, such as to flight vehicles and wing-body configurations subject to interference [8], without consideration of the validity of the method. In these cases, the results obtained from piston theory are notably less accurate [8].

The purpose of the present paper is to briefly review the mathematical basis of piston theory and to use this basis to highlight pitfalls in the application of piston theory to flight vehicles with interfering surfaces. Consideration is also given to the basis for local piston theory and for possible extensions to its application.

2 Theoretical Basis of Piston Theory

2.1 Historical Developments

2.1.1 The Unsteady Analogy

A number of significant milestones in the development of piston theory may be identified. The starting point may be considered to be Hayes' [9] hypersonic similitude for slender bodies, in which the equivalence between a steady three-dimensional hypersonic flow and an unsteady two-dimensional flow in planes perpendicular to the body axis is treated. The concept is also commonly known by the terms "hypersonic equivalence principle", "law of plane sections", and "unsteady analogy" -- the latter is used in this paper. The unsteady analogy forms the basis for a whole class of aerodynamic methods, including the special case of piston theory. In the case of planar flows, the unsteady analogy directly yields piston theory in the form of a set of equations for unsteady gas flow in one dimension. For three-dimensional flows, further assumptions are required to yield the point-function relation.

The unsteady analogy was originally developed for slender bodies at low incidences [3, 9] to the freestream. The extension of the unsteady analogy to large incidences was accomplished by Sychev [10]. The assumptions in Sychev's analysis include smallness of the largest lateral dimension of the body in comparison to its length and a hypersonic cross-flow Mach number. A significant element of Sychev's formulation is that no assumption of small disturbances is required. The resulting set of equations of motion are for two-dimensional unsteady flow in planes perpendicular to the body axis.

The relations between various hypersonic similitudes, including low-incidence small-disturbance theory such of Il'yushin [3], large-incidence similitude due to Sychev [10], strip theory, and piston theory, were summarized by Hayes and Probstein [11] with an estimate of the orders of the error of each theory. The validity of Sychev's [10] method over a greater range of parameters than it was initially restricted to was shown by Voevodenko and Panteleev [12].

2.1.2 Classical Piston Theory

Hayes' [9] conclusions regarding steady three-dimensional flows served as the basis for Lighthill [2] to apply the unsteady analogy to oscillating airfoils, with Lighthill introducing the use of the pressure equation for simple waves and the use of the term "piston". The treatment adopted by Lighthill was based on physical reasoning and is sparse on mathematical development.

An independent analysis for slender bodies at high supersonic speeds was developed by Il'yushin [3], with a detailed mathematical development from the three-dimensional Euler equations of motion and with explicit treatment of the relative orders of magnitude of various terms. The conditions under which the equations reduce to unsteady one-dimensional flow for a wing were given and linear piston theory was derived for the case of small piston Mach numbers. Il'yushin [3] also formally developed the unsteady analogy with a treatment of the order of the error. In the developments of both Lighthill [2] and Il'yushin [3], it was assumed not only that the bodies were slender in both lateral directions, but that the inclination of the bodies to the freestream was small.

Another milestone in the history of piston theory was its application to cylindrical shells, as detailed by Krumhaar [13]. An important assumption employed is that the freestream flow is along the axis of the cylinder (i.e., no cross-flow component exists). An asymptotic development accounting for higher-order contributions due to the curvature of deformations was given, and the consequent "curvature correction term" has become standard in the application of piston theory to panel flutter investigations. The analysis was developed from the potential flow equations.

The aforementioned formulations of piston theory have been label "classical piston theory", and share the common trait that piston theory is used to model perturbations relative to an undisturbed freestream. The use of the simple-wave pressure equation requires that the piston velocity not exceed the speed of sound of the cylinder reference conditions (which in classical piston theory correspond to freestream conditions). This effectively limits the validity of the formulation to a restrictive range of moderately high Mach numbers and small surface slopes; a numerical treatment of this range is given in [14]. The similarity of classical piston theory to other analytical methods is also briefly reviewed in [14]. While the resulting pressure equations are similar, the physical basis from which the equations are derived varies between the methods. Examples of point-function relations for pressure include the formulations of Donov [15] (developed from the method of characteristics) and Van Dyke [16] (based in potential flows). Although they are no longer point-function relations for pressure or based in the unsteady analogy, the "extended piston theories" due to Landahl [17] and Dowell and Bliss [5] are also noted, which account for upstream influence in potential flows. These are called "extended piston theory" because the first terms in the expansions in these methods correspond to the terms in classical piston theory.

2.1.3 Local Piston Theory

The use of the spatially-varying local flow-conditions of an established flow as the cylinder reference conditions in applying piston theory was first suggested by Morgan et al [18], with the local conditions for an airfoil calculated from the oblique shock relations. No mathematical description of the local piston theory formulation was given, and the basis for the method was in physical reasoning.

McIntosh [19] investigated the effect of the reflection from the bow shock of acoustic waves generated by unsteady motion of a wedge from the basis of the hypersonic small-disturbance equations. By considering the uniform steady flow between the wedge and the bow shock as an equivalent freestream for a local piston theory, McIntosh accounted for the effect of the bow shock and the thickness of the wedge, and showed that the resulting local piston theory corresponded exactly with the hypersonic small-disturbance solution for the case of zero reflection of acoustic waves from the bow shock. The definition of the cylinder orientation as perpendicular to the steady surface of the wedge and the definition of the downwash as resulting from perturbations relative to the mean steady flow is mathematically consistent with the definition of local piston theory used in this paper. McIntosh showed that the local piston theory gave consistently and notably better agreement in the unsteady forces with a full solution of the hypersonic small-disturbance equations than was obtained using classical piston theory.

Yates and Bennett [20] used local piston theory applied relative to conditions given by shock-expansion theory, including a second-order local-flow treatment based on Van Dyke's [16] second-order theory. The second-order local-flow model showed better agreement in trends with experimental methods compared to first-order local piston theory as angle-of-attack was increased.

The use of local piston theory to solve for perturbations relative to an inviscid solution from computational fluid dynamics was demonstrated by Hunter [2] and was more broadly introduced into literature by Zhang et al [6]. The resurgence in interest in piston theory -- in particular in local piston theory -- is largely due to the associated reduction in computational cost relative to full unsteady Euler solutions. However, despite the renewed interest in local piston theory, its mathematical basis has not received attention.

3 Mathematical Basis of Piston Theory

As noted in the previous section, piston theory may be developed from the Euler equations by route of the unsteady analogy. The unsteady analogy requires that gradients in flow variables in the axial direction be negligible relative to transverse gradients; in the case of a planar flow, this removal of the axial velocity component as a variable from the Euler equations directly yields the unsteady one-dimensional set of equations associated with piston theory, after a Galilean transform in the axial direction. The reduction to a one-dimensional problem from a three-dimensional flow requires the additional assumption that gradients in one of the lateral directions, as well as gradients in the axial direction, be negligible relative to gradients in the remaining lateral direction. This is illustrated in Figure 1: piston theory requires that both axial as well as span-wise or circumferential gradients be negligible relative to the radial or transverse gradients, equivalent to simultaneously applying the unsteady analogy and strip theory. The assumptions required in the reduction of the dimension of the equations is demonstrated in the following sections.

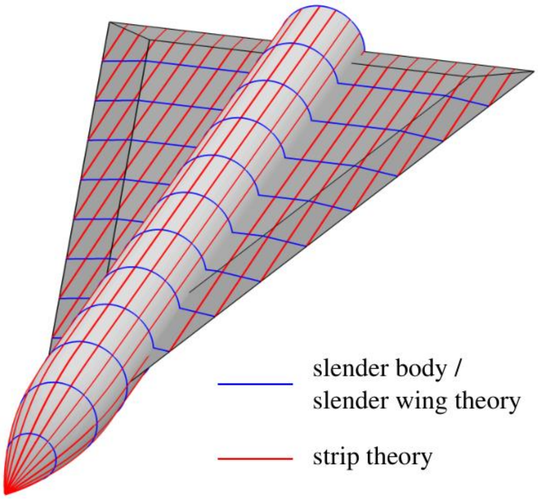

Figure 1: Planes normal to which flow gradients are neglected [22].

3.1 Cartesian Coordinates

The non-dimensional Euler equations in a Cartesian system may be written in general form as

Continuity: $$\left(\frac{U_1}{L_1}\right)\frac{\partial \rho_N u_1}{\partial \xi_1}+\left(\frac{U_2}{L_2}\right)\frac{\partial \rho_N u_2}{\partial \xi_2}+\left(\frac{U_3}{L_3}\right)\frac{\partial \rho_N u_3}{\partial \xi_3}=0 \tag{1}$$

Momentum: $$(\vec{V}\cdot\nabla)u_1=-\left(\frac{P_R}{\rho_R U_1 L_1}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_1} \tag{2a}$$

$$(\vec{V}\cdot\nabla)u_2=-\left(\frac{P_R}{\rho_R U_2 L_2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_2} \tag{2b}$$

$$(\vec{V}\cdot\nabla)u_3=-\left(\frac{P_R}{\rho_R U_3 L_3}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_3} \tag{2c}$$

Energy: $$(\vec{V}\cdot\nabla)i_N=-\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\nabla\cdot\vec{V} \tag{3}$$

Body B.C.: $$\vec{V}\cdot\vec{n}=U_1\eta_1 u_1+U_2\eta_2 u_2+U_3\eta_3 u_3=0 \tag{4}$$

Advection: $$\vec{V}\cdot\nabla\equiv\left(\frac{U_1}{L_1}\right)u_1\frac{\partial}{\partial \xi_1}+\left(\frac{U_2}{L_2}\right)u_2\frac{\partial}{\partial \xi_2}+\left(\frac{U_3}{L_3}\right)u_3\frac{\partial}{\partial \xi_3} \tag{5}$$

Divergence: $$\nabla\cdot\vec{V}=\left(\frac{U_1}{L_1}\right)\frac{\partial u_1}{\partial \xi_1}+\left(\frac{U_2}{L_2}\right)\frac{\partial u_2}{\partial \xi_2}+\left(\frac{U_3}{L_3}\right)\frac{\partial u_3}{\partial \xi_3} \tag{6}$$

where L_1, L_2, L_3 denote the reference lengths for non-dimensional coordinates ξ_1, ξ_2, ξ_3; U_1, U_2, U_3 denote the reference velocities for non-dimensional velocities u_1, u_2,u_3;P_R, ρ_R, and i_R denote the reference values for the non-dimensional pressure P_N, density ρ_N, and internal energy i_N, respectively. The reference values are chosen such that all non-dimensional terms and derivatives are $O(1)$. Reduction of order of the governing equations involves the decoupling of the velocity in the neglected direction from the governing equations through an assumption of relative smallness of dimensionless groups and perturbations to the dimensionless velocities. This will be demonstrated first for the reduction from three dimensions to two, with the axial direction being represented by ξ_1.

3.1.1 Elimination of Axial Dependence

Let the dimensionless velocity u_1 in the axial direction be perturbed by a dimensionless perturbation u_1^*, with $u_1^* = O(1)$, such that $u_1 = 1 + \varepsilon_1 u_1^*$. Introducing the perturbation and rearranging the equations, we have

Continuity: $$k_1 \varepsilon_1 \frac{\partial \rho_N u_1^*}{\partial \xi_1} + k_1 \frac{\partial \rho_N}{\partial \xi_1} + k_2 \frac{\partial \rho_N u_2}{\partial \xi_2} + \frac{\partial \rho_N u_3}{\partial \xi_3} = 0 \tag{7}$$

Body B.C.: $$\left(\frac{U_1 \eta_1}{U_3 \eta_3}\right) \varepsilon_1 u_1^* + \left(\frac{U_1 \eta_1}{U_3 \eta_3}\right) + \left(\frac{U_2 \eta_2}{U_3 \eta_3}\right) u_2 + u_3 = 0 \tag{8}$$

Advection: $$\vec{V} \cdot \nabla = \left(\frac{U_3}{L_3}\right) \left[k_1 \varepsilon_1 u_1^* \frac{\partial}{\partial \xi_1} + k_1 \frac{\partial}{\partial \xi_1} + k_2 u_2 \frac{\partial}{\partial \xi_2} + u_3 \frac{\partial}{\partial \xi_3}\right] \tag{9}$$

Divergence: $$\nabla \cdot \vec{V} = \left(\frac{U_3}{L_3}\right) \left[k_1 \varepsilon_1 \frac{\partial u_1^*}{\partial \xi_1} + k_2 \frac{\partial u_2}{\partial \xi_2} + \frac{\partial u_3}{\partial \xi_3}\right] \tag{10}$$

k_1: $$k_1 = \frac{U_1 L_3}{U_3 L_1} \tag{11}$$

k_2: $$k_2 = \frac{U_2 L_3}{U_3 L_2} \tag{12}$$

It is noted that the components η_1, η_2, η_3 of the surface-normal unit vector $\vec{n}$ may be obtained from the gradient of the equation defining the body surface; this is not developed further at present. In order to decouple u_1 from the equations of motion, it is required that the order of magnitude of the terms involving u_1^* be negligible compared to the terms retained. Equivalently, the equations may be developed up to a certain order which truncates the terms associated with u_1^*. From the above equations, this implies that

$$\varepsilon_1 \ll 1 \tag{12a}$$

$$k_1 \varepsilon_1 \ll 1 \tag{12b}$$

$$\frac{U_1 \eta_1}{U_3 \eta_3} \varepsilon_1 \ll 1 \tag{12c}$$

$$k_1 \varepsilon_1 \ll k_2 \tag{12d}$$

$$\frac{U_1 \eta_1}{U_2 \eta_2} \varepsilon_1 \ll 1 \tag{12e}$$

With these requirements met, the following reduced form of the equations is obtained:

Continuity: $$k_1 \frac{\partial \rho_N}{\partial \xi_1} + k_2 \frac{\partial \rho_N u_2}{\partial \xi_2} + \frac{\partial \rho_N u_3}{\partial \xi_3} = 0 \tag{13}$$

Body B.C.: $$\left(\frac{U_1\eta_1}{U_3\eta_3}\right)+\left(\frac{U_2\eta_2}{U_3\eta_3}\right)u_2+u_3=0 \tag{14}$$

Advection: $$\vec{V}\cdot\nabla=\left(\frac{U_3}{L_3}\right)\left[k_1\frac{\partial}{\partial\xi_1}+k_2u_2\frac{\partial}{\partial\xi_2}+u_3\frac{\partial}{\partial\xi_3}\right] \tag{15}$$

Divergence: $$\nabla\cdot\vec{V}=\left(\frac{U_3}{L_3}\right)\left[k_2\frac{\partial u_2}{\partial\xi_2}+\frac{\partial u_3}{\partial\xi_3}\right] \tag{16}$$

The above may be substituted into Equations (2) through (3), with Equation (2a) becoming redundant. The Galilean transformation using

$$\frac{\partial}{\partial\xi_1}=\frac{L_1}{U_1}\frac{\partial}{\partial t} \tag{17}$$

completes the reduction of the three-dimensional steady problem to a two-dimensional unsteady problem. The physical interpretation of the formulation is that of the body representing a two-dimensional piston driving flow in the transverse plane. The general non-dimensional equations for the boundary conditions at the shock required to close the solution are not treated here, as the additional complexity does not further the illustration of the principle being demonstrated.

For the purpose of discussion, let the ξ_2 represent the spanwise direction and ξ_3 represent the thickness direction of a wing. Equations (12a--c) give the conditions for terms associated with axial perturbation velocities and their gradients to be neglected relative to perturbations in the ξ_3 direction; Equations (12d--e) are for neglecting axial perturbations relative to transverse perturbations in the ξ_2 direction, which is not necessary for our purposes. Consider a slender wing, such that $\eta_1\approx\eta_2\approx\tau$ and $\eta_3\approx 1$, with a thickness ratio of $\tau\ll 1$. The conditions required to satisfy Equations (12a--c) up to order $O(\tau^2)$ for flows described by the Sychev [10] moderate-to-high incidence non-dimensionalization of $U_1/U_3=\cot\alpha$ and by the classical small-incidence non-dimensionalization of $U_1/U_3=1/\tau$ are given in Table 1.

Table 1: Dimensionless groups in the two-dimensional unsteady analogy for slender wings.

Ratio	$U_1/U_3=\cot\alpha$	$U_1/U_3=O(1/\tau)$
ε_1 required for $O(\tau^2)$ accuracy	$O(\tau)$	$O(\tau^2)$
L_3/L_1 required for $O(\tau^2)$ accuracy	$O(\tau)$	$O(\tau)$
k_1	$\tau\cot\alpha$	$O(1)$

Treatment of the lateral perturbations in the reduction from the three-dimensional problem to one dimension requires distinction between two cases. In the first case, one may consider perturbations of the form $u_2=1+\varepsilon_2u_2^*$, in which the perturbation is relative to an established spanwise flow component with an order of magnitude of U_2, as in the case of a steady sideslip. In the second case, the perturbation may be considered relative to a condition of without an established spanwise flow, leading to $u_2=u_2^*$. In this case, the reference value U_2 should be of the order of magnitude of the perturbation u_2^* so that $u_2^*=O(1)$.

3.1.2 Elimination of Lateral Dependence with Established Spanwise Flow

The first case, for which $u_2=1+\varepsilon_2u_2^*$, yields the following set of equations

Continuity: $$k_1\varepsilon_1\frac{\partial\rho_Nu_1^*}{\partial\xi_1}+k_2\varepsilon_2\frac{\partial\rho_Nu_2^*}{\partial\xi_2}+k_1\frac{\partial\rho_N}{\partial\xi_1}+k_2\frac{\partial\rho_N}{\partial\xi_2}+\frac{\partial\rho_Nu_3}{\partial\xi_3}=0 \tag{18}$$

Body B.C.: $$\left(\frac{U_1\eta_1}{U_3\eta_3}\right)\varepsilon_1 u_1^* + \left(\frac{U_2\eta_2}{U_3\eta_3}\right)\varepsilon_2 u_2^* + \left(\frac{U_1\eta_1}{U_3\eta_3}\right) + \left(\frac{U_2\eta_2}{U_3\eta_3}\right) + u_3 = 0 \quad (19)$$

Advection: $$\vec{V}\cdot\nabla = \left(\frac{U_3}{L_3}\right)\left[k_1\varepsilon_1 u_1^*\frac{\partial}{\partial\xi_1} + k_2\varepsilon_2 u_2^*\frac{\partial}{\partial\xi_2} + k_1\frac{\partial}{\partial\xi_1} + k_2\frac{\partial}{\partial\xi_2} + u_3\frac{\partial}{\partial\xi_3}\right] \quad (20)$$

Divergence: $$\nabla\cdot\vec{V} = \left(\frac{U_3}{L_3}\right)\left[k_1\varepsilon_1\frac{\partial u_1^*}{\partial\xi_1} + k_2\varepsilon_2\frac{\partial u_2^*}{\partial\xi_2} + \frac{\partial u_3}{\partial\xi_3}\right] \quad (21)$$

Decoupling u_1 and u_2 from the equations requires that both terms involving u_1^* and terms in u_2^* must be discarded in favour of the remaining terms. This leads to the following requirements in addition to Equations (12a--c):

$$\varepsilon_2 \ll 1 \quad (22a)$$

$$k_2\varepsilon_2 \ll 1 \quad (22b)$$

$$\frac{U_2\eta_2}{U_3\eta_3}\varepsilon_2 \ll 1 \quad (22c)$$

The resulting one-dimensional set of equations that results for the perturbations described by $u_1 = 1 + \varepsilon_1 u_1^*$ and $u_2 = 1 + \varepsilon_2 u_2^*$ is given by

Continuity: $$k_1\frac{\partial\rho_N}{\partial\xi_1} + k_2\frac{\partial\rho_N}{\partial\xi_2} + \frac{\partial\rho_N u_3}{\partial\xi_3} = 0 \quad (23)$$

Momentum: $$\left[k_1\frac{\partial}{\partial\xi_1} + k_2\frac{\partial}{\partial\xi_2} + u_3\frac{\partial}{\partial\xi_3}\right]u_3 = -\left(\frac{P_R}{\rho_R U_3^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial\xi_3} \quad (24)$$

Energy: $$\left[k_1\frac{\partial}{\partial\xi_1} + k_2\frac{\partial}{\partial\xi_2} + u_3\frac{\partial}{\partial\xi_3}\right]i_N = -\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\frac{\partial u_3}{\partial\xi_3} \quad (25)$$

Body B.C.: $$\left(\frac{U_1\eta_1}{U_3\eta_3}\right) + \left(\frac{U_2\eta_2}{U_3\eta_3}\right) + u_3 = 0 \quad (26)$$

The restrictions on the magnitude ε_2 of the spanwise perturbations set by the conditions given by Equations (22a--c), with the previous assumption that $\eta_2 \approx \tau$ and $\eta_3 \approx 1$ for the wing holds, are given in Table 1.

Table 1: Dimensionless groups in the one-dimensional unsteady analogy for slender wings with spanwise flow.

ε_2 required for $O(\tau^2)$ accuracy	$U_2/U_3 = O(\tau)$	$U_2/U_3 = O(1)$
$L_3/L_2 = O(\tau)$	$\varepsilon_2 = O(1), \quad k_2 = O(\tau^2)$	$\varepsilon_2 = O(\tau), \quad k_2 = O(\tau)$
$L_3/L_2 = O(1)$	$\varepsilon_2 = O(\tau), \quad k_2 = O(\tau)$	$\varepsilon_2 = O(\tau^2), \quad k_2 = O(1)$

In each of the scenarios listed in Table 1, the perturbations in the spanwise direction must be no larger than the perturbations in the axial direction in order for the spanwise dependence to be eliminated.

3.1.3 Elimination of Lateral Dependence for No Established Spanwise Flow

The second case, for which $u_2 = u_2^*$, yields the following set of equations

Continuity: $$k_1\varepsilon_1\frac{\partial\rho_N u_1^*}{\partial\xi_1}+k_1\frac{\partial\rho_N}{\partial\xi_1}+k_2\frac{\partial\rho_N u_2^*}{\partial\xi_2}+\frac{\partial\rho_N u_3}{\partial\xi_3}=0 \tag{27}$$

Body B.C.: $$\left(\frac{U_1\eta_1}{U_3\eta_3}\right)\varepsilon_1 u_1^*+\left(\frac{U_1\eta_1}{U_3\eta_3}\right)+\left(\frac{U_2\eta_2}{U_3\eta_3}\right)u_2^*+u_3=0 \tag{28}$$

Advection: $$\vec{V}\cdot\nabla=\left(\frac{U_3}{L_3}\right)\left[k_1\varepsilon_1 u_1^*\frac{\partial}{\partial\xi_1}+k_1\frac{\partial}{\partial\xi_1}+k_2 u_2^*\frac{\partial}{\partial\xi_2}+u_3\frac{\partial}{\partial\xi_3}\right] \tag{29}$$

Divergence: $$\nabla\cdot\vec{V}=\left(\frac{U_3}{L_3}\right)\left[k_1\varepsilon_1\frac{\partial u_1^*}{\partial\xi_1}+k_2\frac{\partial u_2^*}{\partial\xi_2}+\frac{\partial u_3}{\partial\xi_3}\right] \tag{30}$$

Decoupling u_1 and u_2 from the equations requires that both terms involving u_1^* and terms in u_2^* must be discarded in favour of the remaining terms. This leads to the following requirements in addition to Equations (12a--c):

$$k_2 \ll 1 \tag{31a}$$

$$\frac{U_2\eta_2}{U_3\eta_3}\ll 1 \tag{31b}$$

The resulting one-dimensional set of equations that results for the perturbations described by $u_1=1+\varepsilon_1 u_1^*$ and $u_2=u_2^*$ is given by

Continuity: $$k_1\frac{\partial\rho_N}{\partial\xi_1}+\frac{\partial\rho_N u_3}{\partial\xi_3}=0 \tag{32}$$

Momentum: $$\left[k_1\frac{\partial}{\partial\xi_1}+u_3\frac{\partial}{\partial\xi_3}\right]u_3=-\left(\frac{P_R}{\rho_R U_3^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial\xi_3} \tag{33}$$

Energy: $$\left[k_1\frac{\partial}{\partial\xi_1}+u_3\frac{\partial}{\partial\xi_3}\right]i_N=-\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\frac{\partial u_3}{\partial\xi_3} \tag{34}$$

Body B.C.: $$\left(\frac{U_1\eta_1}{U_3\eta_3}\right)+u_3=0 \tag{35}$$

If the assumption of $\eta_2\approx\tau$ and $\eta_3\approx 1$ holds, then in order for Equation (31b) to be satisfied to $O(\tau^2)$, the restriction that $U_2/U_3=O(\tau)$ is required. This implies that the spanwise perturbations must be smaller than the thickness perturbations, which may hold on the centerline of the wing, but may not be expected in the region of wing tips. Transferring this requirement on U_2/U_3 onto Equation (31a) results in the further condition that $L_3/L_2=O(\tau)$, which again suggests that the gradients in the spanwise direction must be smaller than in the thickness direction.

If the assumptions that $L_2/L_1=O(\lambda)$, where λ is the aspect ratio of the wing, and $L_3/L_1=O(\tau)$ hold, then L_3/L_2 may be written as $L_3/L_2=O(\tau/\lambda)$. In this case, it may be seen that Equation (31a) will be satisfied with $U_2/U_3=O(\tau)$ for aspect ratios as low as $\lambda=O(1)$. The validity of the unsteady analogy for these aspect ratios has been commented on by Voevodenko and Panteleev [12], who noted that the accuracy of the unsteady analogy increases for delta wings of lower sweep angles, as the increased leading-edge normal Mach number leads to a better fulfillment of the requirement that $L_3/L_1=O(\tau)$. Lastly, it is noted that the conditions required in discussion produced here are not met where spanwise gradients and perturbation velocities are of the same order as the thickness perturbations and gradients, as may be expected in the region of the wing tips and in regions with significant spanwise flow induced by vortices or body interference.

3.2 Cylindrical Coordinates

Following the nomenclature of the previous section, the non-dimensional Euler equations in a cylindrical system may be written in general form as

Continuity:
$$\left(\frac{U_r}{L_r}\right)\frac{\partial \rho_N u_r}{\partial \xi_r}+\left(\frac{U_r}{L_r}\right)\frac{\rho_N u_r}{\xi_r}+\left(\frac{U_\theta}{L_r L_\theta}\right)\frac{1}{\xi_r}\frac{\partial \rho_N u_\theta}{\partial \xi_\theta}+\left(\frac{U_x}{L_x}\right)\frac{\partial \rho_N u_x}{\partial \xi_x}=0 \tag{36}$$

Momentum:
$$(\vec{V}\cdot\nabla)u_r-\left(\frac{U_\theta^2}{U_r L_r}\right)\frac{u_\theta^2}{\xi_r}=-\left(\frac{P_R}{\rho_R U_r L_r}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_r} \tag{36a}$$

$$(\vec{V}\cdot\nabla)u_\theta+\left(\frac{U_r}{L_r}\right)\frac{u_r u_\theta}{\xi_r}=-\left(\frac{P_R}{\rho_R U_\theta L_r L_\theta}\right)\frac{1}{\rho_N \xi_r}\frac{\partial P_N}{\partial \xi_\theta} \tag{36b}$$

$$(\vec{V}\cdot\nabla)u_x=-\left(\frac{P_R}{\rho_R U_x L_x}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_x} \tag{36c}$$

Energy:
$$(\vec{V}\cdot\nabla)i_N=-\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\nabla\cdot\vec{V} \tag{37}$$

Body B.C.:
$$\vec{V}\cdot\vec{n}=U_r\eta_r u_r+U_\theta\eta_\theta u_\theta+U_x\eta_x u_x=0 \tag{38}$$

Advection:
$$\vec{V}\cdot\nabla\equiv\left(\frac{U_r}{L_r}\right)u_r\frac{\partial}{\partial \xi_r}+\left(\frac{U_\theta}{L_r L_\theta}\right)\frac{u_\theta}{\xi_r}\frac{\partial}{\partial \xi_\theta}+\left(\frac{U_x}{L_x}\right)u_x\frac{\partial}{\partial \xi_x} \tag{39}$$

Divergence:
$$\nabla\cdot\vec{V}=\left(\frac{U_r}{L_r}\right)\frac{\partial u_r}{\partial \xi_r}+\left(\frac{U_r}{L_r}\right)\frac{u_r}{\xi_r}+\left(\frac{U_\theta}{L_r L_\theta}\right)\frac{1}{\xi_r}\frac{\partial u_\theta}{\partial \xi_\theta}+\left(\frac{U_x}{L_x}\right)\frac{\partial u_x}{\partial \xi_x} \tag{40}$$

where the x coordinate is directed along the axis of the body, and L_θ is introduced in order to ensure that gradients in the circumferential (θ) direction are $O(1)$; it may be noted that L_θ always occurs as the product $L_r L_\theta$, so that when radial and circumferential gradients are of the same order, $L_\theta = O(1)$.

3.2.1 Elimination of Axial Dependence

The same procedure is followed for the reduction of the order of the equations. Let the dimensionless velocity u_x in the axial direction be perturbed by a dimensionless perturbation u_x^*, with $u_x^* = O(1)$, such that $u_x = 1 + \varepsilon_x u_x^*$. Introducing the perturbation and rearranging the equations, we have

Continuity:
$$\frac{\partial \rho_N u_r}{\partial \xi_r}+\frac{\rho_N u_r}{\xi_r}+\frac{k_\theta}{\xi_r}\frac{\partial \rho_N u_\theta}{\partial \xi_\theta}+k_x\frac{\partial \rho_N}{\partial \xi_x}+k_x\varepsilon_x\frac{\partial \rho_N u_x^*}{\partial \xi_x}=0 \tag{41}$$

Momentum
$$(\vec{V}\cdot\hat{\nabla})u_r-\left(k_\theta^2 L_\theta^2\right)\frac{u_\theta^2}{\xi_r}=-\left(\frac{P_R}{\rho_R U_r^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_r} \tag{42a}$$

$$(\vec{V}\cdot\hat{\nabla})u_\theta+\frac{u_r u_\theta}{\xi_r}=-\left(\frac{P_R}{\rho_R U_\theta U_r L_\theta}\right)\frac{1}{\rho_N \xi_r}\frac{\partial P_N}{\partial \xi_\theta} \tag{42b}$$

$$(\vec{V}\cdot\hat{\nabla})u_x=-\left(\frac{P_R}{\rho_R U_x U_x}\frac{L_r}{L_x}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_x} \tag{42c}$$

Energy
$$(\vec{V}\cdot\hat{\nabla})i_N=-\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\hat{\nabla}\cdot\vec{V} \tag{43}$$

Body B.C.:
$$u_r+\left(\frac{U_\theta\eta_\theta}{U_r\eta_r}\right)u_\theta+\left(\frac{U_x\eta_x}{U_r\eta_r}\right)+\left(\frac{U_x\eta_x}{U_r\eta_r}\right)\varepsilon_x u_x^*=0 \tag{44}$$

Advection: $$\vec{V}\cdot\hat{\nabla}=\left[u_r\frac{\partial}{\partial\xi_r}+k_\theta\frac{u_\theta}{\xi_r}\frac{\partial}{\partial\xi_\theta}+k_x\frac{\partial}{\partial\xi_x}+k_x\varepsilon_x u_x^*\frac{\partial}{\partial\xi_x}\right] \tag{45}$$

Divergence: $$\hat{\nabla}\cdot\vec{V}=\left[\frac{\partial u_r}{\partial\xi_r}+\frac{u_r}{\xi_r}+\frac{k_\theta}{\xi_r}\frac{\partial u_\theta}{\partial\xi_\theta}+k_x\varepsilon_x\frac{\partial u_x^*}{\partial\xi_x}\right] \tag{46}$$

k_θ: $$k_\theta=\frac{U_\theta}{U_r}\frac{1}{L_\theta} \tag{47}$$

k_x: $$k_x=\frac{U_x L_r}{U_r L_x} \tag{48}$$

In order to decouple u_x from the equations, the dimensionless groups must satisfy the following relations:

$$\varepsilon_x \ll 1 \tag{49a}$$
$$k_x\varepsilon_x \ll 1 \tag{49b}$$
$$\frac{U_x\eta_x}{U_r\eta_r}\varepsilon_x \ll 1 \tag{49c}$$
$$k_x\varepsilon_x \ll k_\theta \tag{49d}$$
$$k_x\varepsilon_x \ll k_\theta^2 L_\theta^2 \tag{49e}$$
$$\frac{U_x\eta_x}{U_\theta\eta_\theta}\varepsilon_x \ll 1 \tag{49f}$$

Once again, the Galilean transformation
$$\frac{\partial}{\partial\xi_x}=\frac{L_x}{U_x}\frac{\partial}{\partial t} \tag{17}$$
completes the reduction of the three-dimensional steady problem to a two-dimensional unsteady problem. The above treatment may be shown to agree with that of both Sychev [10] and Il'yushin [3]. The assumed ratios of reference lengths and velocities used in their analyses are summarized in Table 2. Here, δ is the ratio of the largest transverse dimension of the body to the length of the body, and it is assumed that with the slender body, $\delta \ll 1$.

Table 2: Dimensionless groups in the two-dimensional unsteady analogy for slender bodies.

Ratio	Sychev [10]	Il'yushin [3]
$\frac{L_r}{L_x}\left(\approx\frac{L_r L_\theta}{L_x}\right)$	δ	δ
$\frac{U_x}{U_r}\left(\approx\frac{U_x}{U_\theta}\right)$	$\cot\alpha$	$O\left(\frac{1}{\delta}\right)$
$\frac{\eta_x}{\eta_r}$	$O(\delta)$	$O(\delta)$
k_θ	$O(1)$	$O(1)$
k_x	$\delta\cot\alpha$	$O(1)$
ε_x required for $O(\delta^2)$ accuracy	$O(\delta)$	$O(\delta^2)$

Equations (49a--c) give the conditions for terms associated with axial perturbation velocities and their gradients to be neglected relative to perturbations in the radial direction; Equations (49d--f) are for neglecting axial perturbations relative to transverse perturbations in the circumferential (ξ_θ) direction, which is not necessary for our purposes. The order of magnitude of the terms discarded serves as a measure of the error in applying the unsteady analogy. It is noted that in order to obtain a first-order method, with terms of $O(\delta^2)$ discarded, the axial perturbations must be of $O(\delta^2)$ in the case of low incidence (see Hayes and Probstein [11]) and of $O(\delta)$ in the case of moderate to large incidence (see Sychev [10]). In the latter case, although $\delta \ll 1$, the axial perturbations are not small relative to the transverse velocities. The boundary conditions require that the inclination of the body surface relative to the body axis be of $O(\delta)$, which is the classical assumption of a slender body.

As in the Cartesian case, two perturbation scenarios are considered in treating lateral perturbations. In the first case, an established crossflow is considered with perturbations of the form $u_\theta = 1 + \varepsilon_\theta u_\theta^*$. In the second case, the perturbation is considered relative to a condition with no crossflow, leading to $u_\theta = u_\theta^*$. In this case, the reference value U_θ should be of the order of magnitude of the perturbation u_θ^* so that $u_\theta^* = O(1)$.

3.2.2 Elimination of Lateral Dependence for Established Crossflow

The first case, for which $u_\theta = 1 + \varepsilon_\theta u_\theta^*$, yields the following set of equations

Continuity:
$$\frac{\partial \rho_N u_r}{\partial \xi_r} + \frac{\rho_N u_r}{\xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial \rho_N}{\partial \xi_\theta} + k_x\frac{\partial \rho_N}{\partial \xi_x} + \frac{k_\theta \varepsilon_\theta}{\xi_r}\frac{\partial \rho_N u_\theta^*}{\partial \xi_\theta} + k_x\varepsilon_x\frac{\partial \rho_N u_x^*}{\partial \xi_x} = 0 \tag{50}$$

Momentum
$$(\vec{V}\cdot\hat{\nabla})u_r - \left(\frac{k_\theta^2 L_\theta^2}{\xi_r}\right) - \left(\frac{k_\theta^2 L_\theta^2}{\xi_r}\right)\left[2\varepsilon_\theta u_\theta^* + \varepsilon_\theta^2 u_\theta^{*2}\right] = -\left(\frac{P_R}{\rho_R U_r^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_r} \tag{51a}$$

$$(\vec{V}\cdot\hat{\nabla})\varepsilon_\theta u_\theta^* + \frac{u_r}{\xi_r} + \frac{\varepsilon_\theta u_r u_\theta^*}{\xi_r} = -\left(\frac{P_R}{\rho_R U_\theta U_r L_\theta}\right)\frac{1}{\rho_N \xi_r}\frac{\partial P_N}{\partial \xi_\theta} \tag{51b}$$

$$(\vec{V}\cdot\hat{\nabla})u_x = -\left(\frac{P_R}{\rho_R U_x U_x}\frac{L_r}{L_x}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_x} \tag{51c}$$

Energy
$$(\vec{V}\cdot\hat{\nabla})i_N = -\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\hat{\nabla}\cdot\vec{V} \tag{52}$$

Body B.C.:
$$u_r + \left(\frac{U_\theta \eta_\theta}{U_r \eta_r}\right) + \left(\frac{U_x \eta_x}{U_r \eta_r}\right) + \left(\frac{U_\theta \eta_\theta}{U_r \eta_r}\right)\varepsilon_\theta u_\theta^* + \left(\frac{U_x \eta_x}{U_r \eta_r}\right)\varepsilon_x u_x^* = 0 \tag{53}$$

Advection:
$$\vec{V}\cdot\hat{\nabla} = \left[u_r\frac{\partial}{\partial \xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial}{\partial \xi_\theta} + k_x\frac{\partial}{\partial \xi_x} + k_\theta\varepsilon_\theta\frac{u_\theta^*}{\xi_r}\frac{\partial}{\partial \xi_\theta} + k_x\varepsilon_x u_x^*\frac{\partial}{\partial \xi_x}\right] \tag{54}$$

Divergence:
$$\hat{\nabla}\cdot\vec{V} = \left[\frac{\partial u_r}{\partial \xi_r} + \frac{u_r}{\xi_r} + \frac{k_\theta \varepsilon_\theta}{\xi_r}\frac{\partial u_\theta^*}{\partial \xi_\theta} + k_x\varepsilon_x\frac{\partial u_x^*}{\partial \xi_x}\right] \tag{55}$$

Decoupling u_x and u_θ from the equations requires that both terms involving u_x^* and terms in u_θ^* must be discarded in favour of the remaining terms. This leads to the following requirements in addition to Equations (49a--c):

$$\varepsilon_\theta \ll 1 \tag{56a}$$

$$k_\theta \varepsilon_\theta \ll 1 \tag{56b}$$

$$\frac{U_\theta \eta_\theta}{U_r \eta_r}\varepsilon_\theta \ll 1 \tag{56c}$$

The resulting one-dimensional set of equations that results for the perturbations described by $u_r = 1 + \varepsilon_r u_r^*$ and $u_\theta = 1 + \varepsilon_\theta u_\theta^*$ is given by

Continuity:
$$\frac{\partial \rho_N u_r}{\partial \xi_r} + \frac{\rho_N u_r}{\xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial \rho_N}{\partial \xi_\theta} + k_x \frac{\partial \rho_N}{\partial \xi_x} = 0 \tag{57}$$

Momentum:
$$\left[u_r \frac{\partial}{\partial \xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial}{\partial \xi_\theta} + k_x \frac{\partial}{\partial \xi_x}\right] u_r - \left(\frac{k_\theta^2 L_\theta^2}{\xi_r}\right) = -\left(\frac{P_R}{\rho_R U_r^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_r} \tag{58}$$

Energy:
$$\left[u_r \frac{\partial}{\partial \xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial}{\partial \xi_\theta} + k_x \frac{\partial}{\partial \xi_x}\right] i_N = -\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\left[\frac{\partial u_r}{\partial \xi_r} + \frac{u_r}{\xi_r}\right] \tag{59}$$

Body B.C.:
$$u_r + \left(\frac{U_\theta \eta_\theta}{U_r \eta_r}\right) + \left(\frac{U_x \eta_x}{U_r \eta_r}\right) = 0 \tag{60}$$

The conditions under which Equations (56a--c) are met will be are listed in Table 3 for the case of a body of near-circular cross-section, such that $\eta_\theta \ll 1$ and $\eta_r \approx 1$, such that $\eta_\theta/\eta_r = O(\delta)$. For a slender body with near-circular cross-section at incidence, the ratio U_θ/U_r will be $O(1)$ in the absence of shocks or separation from the body surface. Similarly, the ratio $1/L_\theta$ may be expected to be of $O(1)$ for a slender cylindrical body. For these cases, it follows that $k_\theta = O(1)$ and that perturbation velocities in the circumferential direction must be of second-order smallness, with $\varepsilon_\theta = O(\delta^2)$, relative to the reference circumferential velocity.

Table 3: Dimensionless groups in the one-dimensional unsteady analogy for slender bodies with crossflow.

ε_θ required for $O(\delta^2)$ accuracy	$U_\theta/U_r = O(\delta)$	$U_\theta/U_r = O(1)$
$1/L_\theta = O(\delta)$	$\varepsilon_\theta = O(1),\quad k_\theta = O(\delta^2)$	$\varepsilon_\theta = O(\delta),\quad k_\theta = O(\delta)$
$1/L_\theta = O(1)$	$\varepsilon_\theta = O(\delta),\quad k_\theta = O(\delta)$	$\varepsilon_\theta = O(\delta^2),\quad k_\theta = O(1)$

The principles outlined may be used in the consideration of other body shapes; in particular, for larger η_θ, the requirements on the circumferential perturbations ε_θ to neglect circumferential dependence will be more stringent. These changes in body shape will also lead to regions in which L_θ will no longer be $O(1)$, as may be expected where sharp changes in the body radius occurs or at wing-body junctions.

3.2.3 Elimination of Lateral Dependence for Axial Flow

The second case, for which $u_\theta = u_\theta^*$, yields the following set of equations

Continuity:
$$\frac{\partial \rho_N u_r}{\partial \xi_r} + \frac{\rho_N u_r}{\xi_r} + k_x \frac{\partial \rho_N}{\partial \xi_x} + \frac{k_\theta}{\xi_r}\frac{\partial \rho_N u_\theta^*}{\partial \xi_\theta} + k_x \varepsilon_x \frac{\partial \rho_N u_x^*}{\partial \xi_x} = 0 \tag{61}$$

Momentum
$$(\vec{V} \cdot \hat{\nabla}) u_r - k_\theta^2 L_\theta^2 \frac{u_\theta^{*2}}{\xi_r} = -\left(\frac{P_R}{\rho_R U_r^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_r} \tag{62a}$$

$$(\vec{V} \cdot \hat{\nabla}) u_\theta + \frac{u_r u_\theta^*}{\xi_r} = -\left(\frac{P_R}{\rho_R U_\theta U_r L_\theta}\right)\frac{1}{\rho_N \xi_r}\frac{\partial P_N}{\partial \xi_\theta} \tag{62b}$$

$$(\vec{V} \cdot \hat{\nabla}) u_x = -\left(\frac{P_R}{\rho_R U_x U_x}\frac{L_r}{L_x}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial \xi_x} \tag{62c}$$

Energy
$$(\vec{V} \cdot \hat{\nabla}) i_N = -\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\hat{\nabla} \cdot \vec{V} \tag{63}$$

Body B.C.: $$u_r + \left(\frac{U_x\eta_x}{U_r\eta_r}\right) + \left(\frac{U_\theta\eta_\theta}{U_r\eta_r}\right)u_\theta^* + \left(\frac{U_x\eta_x}{U_r\eta_r}\right)\varepsilon_x u_x^* = 0 \tag{64}$$

Advection: $$\vec{V}\cdot\hat{\nabla} = \left[u_r\frac{\partial}{\partial\xi_r} + k_x\frac{\partial}{\partial\xi_x} + k_\theta\frac{u_\theta^*}{\xi_r}\frac{\partial}{\partial\xi_\theta} + k_x\varepsilon_x u_x^*\frac{\partial}{\partial\xi_x}\right] \tag{65}$$

Divergence: $$\hat{\nabla}\cdot\vec{V} = \left[\frac{\partial u_r}{\partial\xi_r} + \frac{u_r}{\xi_r} + \frac{k_\theta}{\xi_r}\frac{\partial u_\theta^*}{\partial\xi_\theta} + k_x\varepsilon_x\frac{\partial u_x^*}{\partial\xi_x}\right] \tag{66}$$

Decoupling u_x and u_θ from the equations requires that both terms involving u_x^* and terms in u_θ^* must be discarded in favour of the remaining terms. This leads to the following requirements in addition to Equations (49a--c):

$$k_\theta \ll 1 \tag{67a}$$

$$k_\theta^2 L_\theta^2 \ll 1 \tag{67b}$$

$$\frac{U_\theta\eta_\theta}{U_r\eta_r} \ll 1 \tag{67c}$$

The resulting one-dimensional set of equations that results for the perturbations described by $u_r = 1 + \varepsilon_r u_r^*$ and $u_\theta = u_\theta^*$ is given by

Continuity: $$\frac{\partial\rho_N u_r}{\partial\xi_r} + \frac{\rho_N u_r}{\xi_r} + k_x\frac{\partial\rho_N}{\partial\xi_x} = 0 \tag{68}$$

Momentum: $$\left[u_r\frac{\partial}{\partial\xi_r} + k_x\frac{\partial}{\partial\xi_x}\right]u_r = -\left(\frac{P_R}{\rho_R U_r^2}\right)\frac{1}{\rho_N}\frac{\partial P_N}{\partial\xi_r} \tag{69}$$

Energy: $$\left[u_r\frac{\partial}{\partial\xi_r} + k_x\frac{\partial}{\partial\xi_x}\right]i_N = -\left(\frac{P_R}{\rho_R i_R}\right)\frac{P_N}{\rho_N}\left[\frac{\partial u_r}{\partial\xi_r} + \frac{u_r}{\xi_r}\right] \tag{70}$$

Body B.C.: $$u_r + \left(\frac{U_x\eta_x}{U_r\eta_r}\right) = 0 \tag{71}$$

If the assumption that $\eta_\theta/\eta_r = O(\delta)$ holds, the restriction from Equation (67c) requires that the magnitude U_θ of circumferential perturbations be small, in particular, that $U_\theta/U_r = O(\delta)$. In turn, Equation (67a) requires that $1/L_\theta = O(\delta)$. These conditions imply that radial perturbations and gradients be of an order δ larger than circumferential gradients, which may be expected only if the surface slopes due to deformation are larger in the radial-axial plane than in the crossflow plane. Essentially, this requires that the ratio of the amplitude of the deformations around the circumference of the body to the wavelength of deformations be of $O(\delta)$.

3.3 Further Considerations

The application of the unsteady analogy in reducing the three-dimensional steady Euler equations to a set of equations in two perturbation velocities is a familiar result. Under a Galilean transformation, the reduced set of equations represent the unsteady flow of a fluid in planes perpendicular to the axial coordinate, driven by a two-dimensional piston corresponding to the body surface, and bounded by the bow shock at the plane under consideration. Flow in a crossflow plane is independent of flow in planes upstream or downstream of it as the crossflow plane is swept down the length of the body. It is easily seen that under this formulation, the flowfield at any given axial station of the body accounts for the upstream influence and flow history, as washed down by a given "slab" of fluid in the crossflow plane.

With this consideration, it is evident that reduction to one perturbation velocity is not a sufficient condition for a point-function relation between the fluid velocity and pressure and the body surface, as an integration from the furthest upstream point to the point under consideration is performed. The formulation of classical piston theory as by Lighthill [2] disregards the effect of the bow shock and all upstream flow history, and assumes that at each point the piston is being driven into an undisturbed fluid with conditions given by the freestream conditions.

In the analysis of applying the unsteady analogy to a two-dimensional problem by Il'yushin [3], the effect of the bow shock is accounted for, and it is shown that for a body for which the surface inclination decreases along the length of the body, expansion waves upstream of a given axial station do not influence the conditions at the body surface under consideration. Thus, the pressure at a given axial station is given by the perturbation relative to the conditions behind the bow shock, and the formulation becomes a point-function relation. However, in the case of compression waves being generated (due to local increases in the surface inclination down the length of the body), the perturbation relative to the conditions immediately upstream of the local axial station must be modeled, and hence upstream influence enters into the formulation.

The modeling of the perturbation as relative to the conditions of the preceding axial station is essentially a local piston theory formulation. In the approach by Il'yushin [3], the solution at each station is marched downstream, with the condition that reflection from the bow shock of waves produced by the piston are neglected. The effect of reflection was considered by McIntosh [19], who showed that the local piston theory formulation corresponded to the exact solution of the reduced-order Euler equations for the special case of no reflection from the bow shock. Accounting for reflection results in the point-function relation being lost, as the propagation and reflection of waves from upstream disturbances must be modeled.

These considerations may be carried further in applying the unsteady analogy to unsteady three-dimensional flows. As noted by Hayes and Probstein [11], the application to unsteady flows requires that the reduced frequency of the unsteady disturbances, based on the axial reference velocity and an axial length scale associated with the disturbance, be small. In the development by Hayes [9], the assumption was made that disturbances at different stations along the body are in phase; the reduced frequency serves as an indication of the phase difference between disturbances over the length of the body, and thus the requirement for small reduced frequency may be interpreted as an assumption of small phase differences in the disturbances. The analysis of McIntosh [19] showed that the reflection from the bow shock of waves produced by piston motion can result in significant phase differences in unsteady pressures over the length of an oscillating body.

In the development of the general equations presented in this paper, the reference values used in the non-dimensionalization have been treated as constants. The values are required to be chosen such that the non-dimensional flow variables and gradients are $O(1)$. The analyses of Il'yushin [3] and Sychev [10] were developed for cases in which a given set of reference values are appropriate throughout the entire flowfield between the body and the bow shock, allowing the unsteady analogy to apply globally in this flow region. Local breakdown of the conditions were noted in the regions of wing tips [3], blunted noses [10], and the leeside flowfield at large incidence [10] without a global breakdown of the unsteady analogy, and it was reasoned that the contribution from the regions in error was small relative to the overall loading on the body. It is suggested that the general equations and conditions treated in the present paper may be used in an asymptotic development in cases where regions in the flowfield exist with important differences in scale, requiring different reference values for the non-dimensionalization.

4 Basis for Further Applications

4.1 Wing-Body Configurations

The basis for piston theory to be applied to wing was treated in Section 3.1 and the basis for application to slender bodies was treated in Section 3.2. The same process may be used in applying the unsteady analogy to wing-body configurations, as the equations are the same. The key difference lies in the change in the flow gradients and perturbations, and the references values used in order to non-dimensionalize them. Figure 2 illustrates regions for a wing-body configuration at incidence in which large lateral gradients may be expected due to interference and flow structures such as vortices.

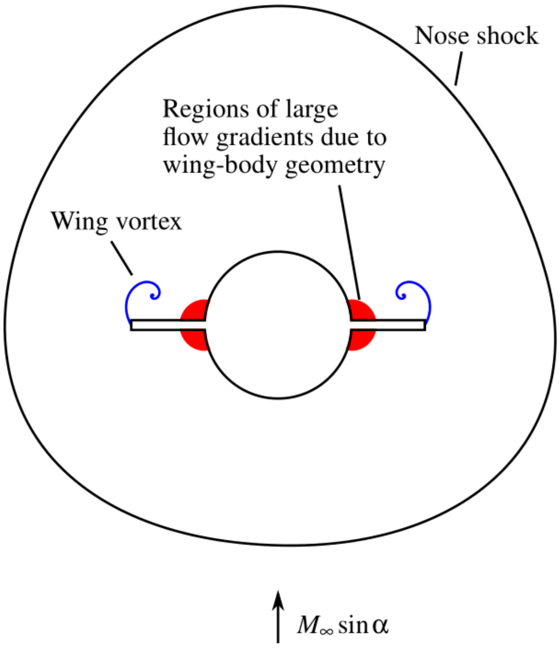

Figure 2: Typical crossflow features at small incidence [22].

As has been discussed in the sections developing the mathematical basis, the reduction of the three-dimensional flow formulation to a one-dimensional formulation is not valid in regions where the ratios of lateral velocities and length scales are both of $O(1)$, and the formulation is two-dimensional in the crossflow plane. If the perturbations to the axial velocity or the local length scale in the axial direction in these regions become large enough that Equations (12a--c) or (49a--c) are no longer satisfied, then the unsteady analogy no longer applies locally. It has been noted [3,10--12] that local failure of the unsteady analogy does not imply a global failure.

Furthermore, the considerations of Section 3.2.2 suggest that the elimination of the circumferential flow variables is not possible when an established crossflow exists for which the ratios of circumferential-to-radial velocities and length scales are of $O(1)$, which is the typical case for a body of near-circular cross-section.

Broadly summarized, piston theory is not valid in the region of wing tips and the wing-body junctions. For configurations of small span-to-diameter ratio, piston theory may not be applicable at incidence over any portion of the wing. The influence of wing vortices may lead to the spanwise gradients on the leeside surface of the wing becoming comparable to the thickness-direction gradients, invalidating the one-dimensional formulation required for piston theory. The neglect of circumferential terms on the body for established crossflow requires perturbations in the circumferential direction to be of second-order smallness relative to the reference crossflow velocities, suggesting that piston theory may not be applied to the body at non-zero incidence.

4.2 Local Piston Theory

The perturbation form of the Euler equations considered in Sections 3.1.1, .3.1.2, 3.2.1 and 3.2.2 may be used in providing a mathematical basis for local piston theory. Extension of the analysis requires a number of changes, namely that

1. The axes should be redefined to be tangential to the local mean steady surface, with directions ξ_3 and ξ_r normal to the surface.
2. The reference values should be redefined to correspond to the local mean steady values being non-dimensionalized.

In this revised formulation, U_3, L_3, U_r, and L_r are used to non-dimensionalize the perturbations, as the local mean steady flow has a zero component in the new ξ_3 or ξ_r directions. The requirements for reduction to a one-dimensional formulation are essentially equivalent to those associated with small-incidence (such as due to Il'yushin [3]), with the lateral perturbations being an order smaller than the local mean steady flow tangential to the surface, and perturbations along the surface being two orders smaller than the mean steady flow.

4.3 Extension to Viscous Flows

The application of the unsteady analogy to hypersonic viscous flows has been established [23,24] in the inviscid flow region around the effective shape (due to boundary layer displacement) of a body. The applicability of the unsteady analogy within the boundary layer region as questionable, due to the typically large gradients and variation in scales. A review of non-dimensionalizations used in analyzing hypersonic flows is given in [24], which notes a number of approaches in which terms with second derivatives in the axial flow direction and the pressure gradient in this direction are neglected, allowing a single set of non-dimensionalization parameters to be used throughout the entire flowfield between the body and the bow shock. The neglect of the pressure gradient in the axial flow direction was estimated [24] to have an effect of the order of 10% on the flow solution. A rigorous approach would require the use of more than one set of non-dimensionalization reference values, leading to two or more sets of non-dimensional governing equations for different regions of the flow which would require matching. In this approach, it may be expected that the unsteady analogy would apply in the inviscid outer region, should it exist, but would breakdown in the viscous inner region.

5 Conclusions

The theoretical and mathematical basis of piston theory as a special case of the unsteady analogy has been reviewed, with the conditions under which the reduction to a one-dimensional set of equations is possible given. Consideration was given to scenarios under which these conditions are not met and for which the neglect of lateral dependence in the equations is not possible. These scenarios include the previously identified regions of large gradients for wings, namely the wing tips and trailing edges, as well as blunted nose of slender bodies. In the present work, consideration was given to the regions for which lateral dependence cannot be neglected when applying the unsteady analogy to wing-body configurations. Regions in which the reduction to piston theory may not be possible include the vicinity of the wing-body junction and, in the case of moderate incidence and large influence of shed vortices, the leeside of the wings. The applicability of piston theory to the body under conditions of crossflow was shown to require that circumferential perturbation velocities be of second-order smallness relative to the established crossflow velocities. The perturbation form of the Euler equations were used to provide a mathematical basis for the application of local piston theory. It was shown that the conditions required for local piston theory to apply are equivalent to the classical hypersonic small-disturbance conditions applied to local perturbation velocities, rather than to perturbations relative to the freestream. The extension of the unsteady analogy to viscous flows was briefly considered, with the validity in regions with large viscous forces and gradients being questionable.

References

[1] Ashley, H., Zartarian, G., Piston Theory --- a New tool for the Aeroelastician, Journal of the Aeronautical Sciences, Vol. 23, No. 12, pp. 1109--1118, 1956.

[2] Lighthill, M. J., Oscillating Airfoils at High Mach Numbers, Journal of the Aeronautical Sciences, Vol. 20, No. 6, pp. 402--406, 1953.

[3] Il'yushin, A. A., The Law of Plane Sections in the Aerodynamics of High Supersonic Speeds, Journal of Applied Mathematics and Mechanics, Vol. 20, No. 6, 1956.

[4] Meijer, M.-C., Dala, L., Generalized Formulation and Review of Piston Theory for Airfoils, AIAA Journal, Vol. 54, No. 1, pp. 17--27, 2016.

[5] Dowell, E. H., Bliss, D. B., New Look at Unsteady Supersonic Potential Flow Aerodynamics and Piston Theory, AIAA Journal, Vol. 51, No. 9, pp. 2278--2281, 2013.

[6] Zhang, W.-W., Ye, Z.-Y, Zhang, C.-A., Liu, F., Supersonic Flutter Based on a Local Piston Theory, AIAA Journal, Vol. 47, No. 10, pp. 2321--2328, 2009.

[7] McNamara, J. J., Crowell, A. R., Friedmann, P. P., Glaz, B., Gogulapati, A., Approximate Modeling of Unsteady Aerodynamics for Hypersonic Aeroelasticity, Journal of Aircraft, Vol. 47, No. 6, pp. 1932--1945, 2010.

[8] Shi, X., Tang, G., Yang, B., Li, H., Supersonic Flutter Analysis of Vehicles at Incidence Based on Local Piston Theory, Journal of Aircraft, Vol. 49, No. 1, pp. 333--337, 2012.

[9] Hayes, W. D., On Hypersonic Similitude, Quarterly of Applied Mathematics, Vol. 5, No. 1, pp. 105--106, 1947.

[10] Sychev, V. V., Three-Dimensional Hypersonic Gas Flow Past Slender Bodies at High Angles of Attack, Journal of Applied Mathematics and Mechanics, Vol. 24, No. 2, pp. 296--306, 1960.

[11] Hayes, W. D., Probstein, R. F., Hypersonic Inviscid Flow, Dover Publications, 2004.

[12] Voevodenko, N. V., Panteleev, I. M., Numerical Modeling of Supersonic Flow Over Wings with Varying Aspect Ratio on a Broad Range of Angles of Attack Using the Law of Plane Sections, Fluid Dynamics, Vol. 27, No. 2, pp. 239--244, 1992.

[13] Krumhaar, H., The Accuracy of Linear Piston Theory when Applied to Cylindrical Shells, AIAA Journal, Vol. 1, No. 6, pp. 1448--1449, 1963.

[14] Meijer, M.-C., Dala, L., On the Validity Range of Piston Theory, 16th International Forum on Aeroelasticity and Structural Dynamics, Paper IFASD-2015-004, St Petersburg, Russia, July 2015.

[15] Donov, A. E., A Flat Wing With Sharp Edges in a Supersonic Stream, NACA TM-1394, 1956.

[16] Van Dyke, M. D., A Study of Second-Order Supersonic Flow Theory, NACA Report 1081, 1952.

[17] Landahl, M. T., Unsteady Flow Around Thin Wings at High Mach Numbers, Journal of the Aeronautical Sciences, Vol 24, No. 1, pp. 33-38, 1957.

[18] Morgan, H. G., Runyan, H. L., Huckel, V., Theoretical Consideratinos of Flutter at High Mach Numbers, Journal of the Aerospace Sciences, Vol. 25, No. 4, pp. 246--258, 1958.

[19] McIntosh, S. C., Hypersonic Flow over an Oscillating Wedge, AIAA Journal, Vol. 3, No. 3, pp. 433--440, 1965.

[20] Yates, E. C., Bennett, R. M., Analysis of Supersonic-Hypersonic Flutter of Lifting Surfaces at Angle of Attack, Journal of Aircraft, Vol. 9, No. 7, pp. 481--489, 1972

[21] Hunter, J. P., An Efficient Method for Time-Marching Supersonic Flutter Predictions using CFD, M.S. Dissertation, Oklahoma State University, Stillwater, OK, 1997.

[22]Meijer, M.-C., Dala, L., Karkle, P. G., On Applying Piston Theory in Interference Flows, 30th Congress of the International Council of the Aeronautical Sciences, Paper ICAS-2016-0225, Daejeon, South Korea, September 2016.

[23]Luniev, V. V., On The Similarity of Hypersonic Viscous Flows Around Slender Bodies, Prikl. Mat. Mekh., Vol. 23, No. 1, pp. 193--197, 1959.

[24]Mikhailov, V. V., Neiland, V. Ya., Sychev, V. V., THe Theory of Viscous Hypersonic Flow, Annual Review of Fluid Mechanics 3, pp. 371--396 , 1971.